On generalised t-designs and their parameters

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Abstract

Recently, P.J. Cameron introduced a new class of block designs which generalises the class of t-designs, and also includes orthogonal arrays, resolved 2-designs, and other classes of combinatorial designs. The main idea is that the set of points of the design is structured by an ordered partition of that set, and the blocks of the design and t-subsets of the point-set also have structure with respect to this point-set partition. There is now the prospect that techniques, theorems and bounds for one class of combinatorial designs may be generalised to many classes of designs at once, via these generalised t-designs. In this paper, we investigate the parameters of generalised t-designs, providing some strong necessary conditions for the existence of certain types of such designs. We show that the t-design constants λ_i (the number of blocks containing an i-subset of the points, where $i \leq t$) and λ_i^j (the number of blocks containing an i-subset I of the points and disjoint from a j-subset J of the points, where $I \cap J = \emptyset$ and $i + j \leq t$) have natural counterparts for generalised t-designs. We also generalise N.S. Mendelsohn's concept of "intersection numbers of order r" for t-designs, and show that analogous equations to those of Mendelsohn hold for generalised t-designs.

[Keywords: block design, t-design, generalised t-design, orthogonal array]

1 Introduction

Recently, P.J. Cameron [1] introduced a new class of block designs which generalises the class of t-designs, and also includes orthogonal arrays, resolved 2-designs, and other classes of combinatorial designs. The main idea is that the set of points of the design is structured by an ordered partition of that set, and the blocks of the design and t-subsets of the point-set also have structure with respect to this point-set partition. There is now the prospect that techniques, theorems and bounds for one class of combinatorial designs may be generalised to many classes of designs at once, via these generalised t-designs. In this paper we investigate the parameters of generalised t-designs, providing some necessary conditions for the existence of certain types of such designs, as well as generalising certain classic results in the theory of t-designs.

Let V be a finite set and $\mathbf{V} = (V_1, \ldots, V_m)$ an ordered partition of V. For S a subset of V, we define the $\mathbf{V}$*-type* of S, denoted $[S]_{\mathbf{V}}$, to be the m-vector of non-negative integers

$$(|S \cap V_1|, \ldots, |S \cap V_m|).$$

When the ordered partition $\mathbf{V}$ is clear from the context, we may just say *type* for $\mathbf{V}$-type, and denote the type of S by $[S]$. (Note that if $\mathbf{V} = (V)$ then the $\mathbf{V}$-type of a subset S of V is simply a vector with a single co-ordinate giving the size of S.) Given any two vectors $\mathbf{r} = (r_1, \ldots, r_n)$, $\mathbf{s} = (s_1, \ldots, s_m)$ of non-negative integers, we write $\mathbf{r} \leq \mathbf{s}$ to mean that $n = m$ and $r_i \leq s_i$ for $i = 1, \ldots, m$. Further, we denote $\sum_{i=1}^{m} s_i$ by $|\mathbf{s}|$, so that, when $S \subseteq V$ and $\mathbf{s} = [S]_{\mathbf{V}}$, we have $|S| = |\mathbf{s}|$.

A *block design* is an ordered pair $(V, \mathcal{B})$, such that V is a finite non-empty set, whose elements are called *points*, and $\mathcal{B}$ is a finite non-empty multiset of subsets of V called *blocks*.

For t a non-negative integer, a t-(v, k, λ) *design* (or simply a t-*design*) is a block design $(V, \mathcal{B})$ satisfying:

- $|V| = v$;
- each block has the same size k, with $k > 0$ and $t \leq k$;
- each t-subset of V is contained in the same (positive) number λ of blocks.

For t a non-negative integer and V a finite non-empty set, a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ *design* (or simply a *generalised t-design*) with *point-set* V is an ordered pair $(\mathbf{V}, \mathcal{B})$, such that $\mathbf{V}$ is an ordered partition of V, $(V, \mathcal{B})$ is a block design, and the following properties hold:

- $[V]_{\mathbf{V}} = \mathbf{v}$;
- each block has the same $\mathbf{V}$-type $\mathbf{k}$, with each entry in $\mathbf{k}$ positive, and $t \leq |\mathbf{k}|$;
- for every tuple $\mathbf{t}$ of non-negative integers satisfying $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$, each t-subset T of V having $[T]_{\mathbf{V}} = \mathbf{t}$ is contained in the same (positive) number $\lambda_{\mathbf{t}}$ of blocks.

We call $t, \mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}})$ the *parameters* of a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design D. The sequence $(\lambda_{\mathbf{t}})$, if explicitly given, is with respect to some fixed total ordering (say lexicographic) of those $\mathbf{t}$ with $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$. We denote by $V(D)$ the point-set of D.

Example 1 Let $D := (\mathbf{V}, \mathcal{B})$, where $\mathbf{V} := (\{1, 2, 3, 4\}, \{5, 6, 7\})$, and

$$\mathcal{B} := [\{1, 2, 5\}, \{3, 4, 5\}, \{1, 3, 6\}, \{2, 4, 6\}, \{1, 4, 7\}, \{2, 3, 7\}].$$

Then $V(D) = \{1, \ldots, 7\}$, and D is a 2-$((4, 3), (2, 1), (\lambda_{(1,1)}, \lambda_{(2,0)}))$ design with $\lambda_{(1,1)} = \lambda_{(2,0)} = 1$.

Note that a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design with $\mathbf{k} = (k)$ is (essentially) the same thing as a t-design with block size k, and a t-$(\mathbf{v}, (1, 1, \ldots, 1), (\lambda_{\mathbf{t}}))$ design is (essentially) the same thing as an orthogonal array of strength t. Many more examples and classes of generalised t-designs are given in [1].

In this paper, we shall show that the t-design constants λ_i (the number of blocks containing an i-subset of the points, where $i \leq t$) and λ_i^j (the number of blocks containing an i-subset I of the points and disjoint from a j-subset J of the points, where $I \cap J = \emptyset$ and $i + j \leq t$) have natural counterparts for generalised t-designs where the size of a set is replaced by its type.

Cameron [1] defines and almost exclusively studies t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ designs with $\lambda_{\mathbf{t}} = \lambda$ (a constant) for all $\mathbf{t}$; these are called t-$(\mathbf{v}, \mathbf{k}, \lambda)$ *designs*. (He also requires $0 < t < |\mathbf{k}|$, which we do not.) Thus, the design D in Example 1 is a 2-$((4, 3), (2, 1), 1)$ design. We shall prove that if we have a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with block size k and $2 \le t \le k-2$, then $\mathbf{k} \in \{\mathbf{v}, (k), (k-1,1), (1,k-1), (1,1,\ldots,1)\}$.

Finally, we shall generalise Mendelsohn's concept of "intersection numbers of order r" for t-designs, and show that analogous equations to those of Mendelsohn [5] hold for generalised t-designs. We remark that, as well as our generalisation of Mendelsohn's equations, the block intersection polynomial techniques of Cameron and Soicher [2, 7] can also be applied to study generalised t-designs with given parameters.

2 Generalising the constants λ_i of a t-design

We start with a very useful result.

Theorem 2.1 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design with $\mathbf{V} = (V_1, \ldots, V_m)$, $\mathbf{v} = (v_1, \ldots, v_m)$, $\mathbf{k} = (k_1, \ldots, k_m)$, and $t > 0$. Let $S \subseteq V(D)$, with $|S| = t-1$ and $\mathbf{s} := [S]_{\mathbf{V}} = (s_1, \ldots, s_m) \le \mathbf{k}$, and let λ_S be the number of blocks of D containing S. Then*

$$\lambda_S = \lambda_{\mathbf{t}_i}(v_i - s_i)/(k_i - s_i)$$

for each i with $k_i > s_i$, where $\mathbf{t}_i = \mathbf{s}(i+) := (s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1}, \ldots, s_m)$. In particular, the number of blocks containing S depends only on the parameters $t, \mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}})$ of D and the $\mathbf{V}$-type $\mathbf{s}$ of S.

Proof. Let $i \in \{1, \ldots, m\}$, such that $k_i > s_i$. We count in two ways the number N of ordered pairs (a, B) such that $a \in V_i \setminus S$ and B is a block containing $S \cup \{a\}$.

We have $v_i - s_i = |V_i \setminus S|$, $\mathbf{t}_i = [S \cup \{a\}]_{\mathbf{V}}$ for each $a \in V_i \setminus S$, and $|\mathbf{t}_i| = t$, so

$$N = (v_i - s_i)\lambda_{\mathbf{t}_i}.$$

On the other hand, each block containing S contains exactly $k_i - s_i$ elements of $V_i \setminus S$, and so

$$N = \lambda_S(k_i - s_i).$$

The result follows. ∎

Corollary 2.2 *Suppose D is a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design and $I \subseteq V(D)$, with $|I| \le t$. Then the number λ_I of blocks containing I is a constant $\lambda_{\mathbf{i}}$, depending only on the parameters of D and the $\mathbf{V}$-type $\mathbf{i}$ of I.*

Proof. If $\mathbf{i} \not\leq \mathbf{k}$ then $\lambda_I = 0$. We now assume $\mathbf{i} \leq \mathbf{k}$ and proceed by induction on $n := t - |I|$.

If $n = 0$ then $|I| = t$, and the required constant is given in the parameters of D.

Suppose $n > 0$. Inductively, for each $J \subseteq V(D)$ with $|J| = |I| + 1$, the number λ_J of blocks containing J is a constant $\lambda_{\mathbf{j}}$, depending only on the parameters of D and the $\mathbf{V}$-type $\mathbf{j}$ of J. Thus, we can apply the theorem above to compute λ_I using only $t, \mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}})$ and $\mathbf{i}$. ∎

Corollary 2.3 *A generalised t-design is a generalised s-design for each $s = 0, \ldots, t$.*

Example 2 Let H be the cyclic group of order 10 generated by

$$(1, 2, 3, 4, 5)(7, 8, 9, 10, 11, 12, 13, 14, 15, 16),$$

and let $\mathcal{B}$ be the union of the H-orbits of $\{1, 2, 4, 7, 10, 11, 13\}$, $\{1, 2, 4, 9, 10, 14, 15\}$, $\{1, 2, 6, 7, 8, 9, 11\}$, and $\{1, 2, 6, 9, 11, 14, 16\}$. The reader can verify that

$$E := ((\{1, \ldots, 6\}, \{7, \ldots, 16\}), \mathcal{B})$$

is a 2-$((6, 10), (3, 4), (\lambda_{(0,2)} = 4, \lambda_{(1,1)} = 6, \lambda_{(2,0)} = 6))$ design. (The design E was found using the function `BlockDesigns` in the DESIGN package [6] for GAP [3]. Indeed, using this function it is easy to classify, up to isomorphism, the 36 H-invariant generalised 2-designs with the same parameters as E.) Now any generalised 2-design with the same parameters as E is also a 1-$((6, 10), (3, 4), (\lambda_{(0,1)} = 12, \lambda_{(1,0)} = 15))$ design, and also a 0-$((6, 10), (3, 4), 30)$ design.

3 Generalising the constants λ_i^j of a t-design

Theorem 3.1 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design, and let $I, J \subseteq V(D)$, $I \cap J = \emptyset$, and $|I \cup J| \leq t$. Then the number λ_I^J of blocks containing I and disjoint from J is a constant $\lambda_{\mathbf{i}}^{\mathbf{j}}$, depending only on the parameters of D and the $\mathbf{V}$-types $\mathbf{i}$ and $\mathbf{j}$ of I and J respectively.*

Proof. We shall use the following notation. For $\mathbf{u} = (u_1, \ldots, u_m)$ an m-tuple of integers, and $\ell \in \{1, \ldots, m\}$, define

$$\mathbf{u}(\ell+) := (u_1, \ldots, u_{\ell-1}, u_\ell + 1, u_{\ell+1}, \ldots, u_m),$$

$$\mathbf{u}(\ell-) := (u_1, \ldots, u_{\ell-1}, u_\ell - 1, u_{\ell+1}, \ldots, u_m).$$

The proof is by induction on $|\mathbf{j}|$, the size of J. If $|\mathbf{j}| = 0$ then $J = \emptyset$ and $\lambda_I^J = \lambda_{\mathbf{i}}$.

Now suppose $|\mathbf{j}| > 0$, $\mathbf{j} = (j_1, \ldots, j_m)$, and $\mathbf{V} = (V_1, \ldots, V_m)$. Choose ℓ such that $j_\ell > 0$ and let $a \in J \cap V_\ell$. Then $a \notin I$, and

$$\lambda_I^J = \lambda_I^{J\setminus\{a\}} - \lambda_{I\cup\{a\}}^{J\setminus\{a\}}.$$

Thus, by induction, for every $\ell \in \{1, \ldots, m\}$ such that $j_\ell > 0$, we have:

$$\lambda_I^J = \lambda_{\mathbf{i}}^{\mathbf{j}(\ell-)} - \lambda_{\mathbf{i}(\ell+)}^{\mathbf{j}(\ell-)}.$$

In particular, λ_I^J depends only on the parameters of D and $\mathbf{i}$ and $\mathbf{j}$. ■

Example 3 Let D be any 2-$((6,10),(3,4),(\lambda_{(0,2)} = 4, \lambda_{(1,1)} = 6, \lambda_{(2,0)} = 6))$ design. Then for such a D, we have $\lambda_{(1,0)}^{(0,1)} = \lambda_{(1,0)}^{(0,0)} - \lambda_{(1,1)}^{(0,0)} = 15 - 6 = 9$.

In analogy to t-$(v, k, 1)$ designs (Steiner systems), we can say more for t-$(\mathbf{v}, \mathbf{k}, 1)$ designs when I and J are disjoint subsets of some block.

Theorem 3.2 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, 1)$ design, and let I and J be disjoint subsets of some block of D. Then the number λ_I^J of blocks containing I and disjoint from J is a constant $\lambda_{\mathbf{i}}^{\mathbf{j}}$, depending only on the parameters of D and the $\mathbf{V}$-types $\mathbf{i}$ and $\mathbf{j}$ of I and J respectively.*

Proof. The proof is by an induction on $|\mathbf{j}|$, similar to the proof of the preceding theorem.

Suppose $|\mathbf{j}| = 0$; that is, $J = \emptyset$. Then $\lambda_I^J = \lambda_{\mathbf{i}}$ if $|\mathbf{i}| \leq t$, and otherwise $\lambda_I^J = 1$.

Now suppose $|\mathbf{j}| > 0$. Then we get the same recursive formulation for λ_I^J as in the proof of the preceding theorem. ■

We remark that the facts that the $\lambda_{\mathbf{i}}$ must be integers and that the $\lambda_{\mathbf{i}}$ and $\lambda_{\mathbf{i}}^{\mathbf{j}}$ can often be computed in more than one way provide necessary conditions on the parameters of a generalised t-design.

4 The block structure of t-$(\mathbf{v}, \mathbf{k}, \lambda)$ designs

In this section, we determine restrictions on the block structure of a generalised t-design with constant $\lambda_{\mathbf{t}}$. First we prove:

Lemma 4.1 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $t > 0$, $\mathbf{v} = (v_1, \ldots, v_m)$, $\mathbf{k} = (k_1, \ldots, k_m)$, block size $k \geq t+1$, and with $k_i = v_i$ for some i. Then $\mathbf{k} = \mathbf{v}$.*

Proof. Without loss of generality, suppose $k_1 = v_1$, and to obtain a contradiction, assume $\mathbf{k} \neq \mathbf{v}$. Then $m > 1$, and we can suppose, without loss of generality, that $k_2 < v_2$. Now take $S \subseteq V$ with $|S| = t - 1$ and $\mathbf{s} := [S]_{\mathbf{V}} = (s_1, \ldots, s_m) \leq \mathbf{k}$, such that $k_1 - s_1 \geq 1$ and $k_2 - s_2 \geq 1$ (this is possible since $t > 0$ and $k \geq t+1$). Now, applying Theorem 2.1, we get that the number of blocks containing S is

$$\lambda_S = \lambda(v_1 - s_1)/(k_1 - s_1) = \lambda,$$

and also

$$\lambda_S = \lambda(v_2 - s_2)/(k_2 - s_2) > \lambda,$$

a contradiction. ∎

Theorem 4.2 *Suppose D is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $t \geq 2$ and block size $k \geq t+2$. Then $\mathbf{k} \in \{\mathbf{v}, (k), (k-1, 1), (1, k-1), (1, 1, \ldots, 1)\}$.*

Proof. Suppose $D = (\mathbf{V}, \mathcal{B})$, $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{k} = (k_1, \ldots, k_m)$. If $m = 1$ then $\mathbf{k} = (k)$, and there is nothing to prove. We shall consider the cases $m = 2$ and $m \geq 3$ separately.

Suppose $m = 2$. If $k_1 = 1$ or $k_2 = 1$ there is nothing to prove, and so we assume that $k_1, k_2 \geq 2$. Now take $U \subseteq V(D)$ with $|U| = t - 2$ and $[U]_{\mathbf{V}} = (u_1, u_2)$, such that $k_1 - u_1 \geq 2$ and $k_2 - u_2 \geq 2$ (this is possible since $t \geq 2$ and $k \geq t+2$). Let $\mathbf{V} = (V_1, V_2)$, let $a_i \in V_i \setminus U$ and let n_i be the number of blocks containing $U \cup \{a_i\}$ ($i = 1, 2$). Then, by Theorem 2.1, we have:

$$n_1 = \lambda(v_1 - (u_1 + 1))/(k_1 - (u_1 + 1)) = \lambda(v_2 - u_2)/(k_2 - u_2),$$

$$n_2 = \lambda(v_1 - u_1)/(k_1 - u_1) = \lambda(v_2 - (u_2 + 1))/(k_2 - (u_2 + 1)).$$

Thus

$$(v_1 - u_1 - 1)(k_2 - u_2) = (v_2 - u_2)(k_1 - u_1 - 1), \tag{1}$$

$$(v_1 - u_1)(k_2 - u_2 - 1) = (v_2 - u_2 - 1)(k_1 - u_1). \tag{2}$$

Subtracting (2) from (1), we obtain

$$-k_2 + u_2 + v_1 - u_1 = -v_2 + u_2 + k_1 - u_1,$$

and so $v_1 + v_2 = k_1 + k_2$, and since $0 \leq k_i \leq v_i$, we must have $k_1 = v_1$ and $k_2 = v_2$; that is, $\mathbf{k} = \mathbf{v}$.

Suppose now $m \geq 3$ and $\mathbf{k} \neq (1, 1, \ldots, 1)$. Without loss of generality, $k_1 \geq 2$. Now take $U \subseteq V$ with $|U| = t-2$ and $[U]_{\mathbf{V}} = (u_1, \ldots, u_m) \leq \mathbf{k}$, such that $k_1 - u_1 \geq 2$, $k_2 - u_2 \geq 1$ and $k_3 - u_3 \geq 1$ (this is possible since $t \geq 2$ and $k \geq t+2$). Let $\mathbf{V} = (V_1, \ldots, V_m)$, let $a_i \in V_i \setminus U$ and let n_i be the number of blocks containing $U \cup \{a_i\}$ $(i = 1, 2, 3)$. Then, by Theorem 2.1, we have:

$$n_1 = \lambda(v_1 - (u_1 + 1))/(k_1 - (u_1 + 1)) = \lambda(v_2 - u_2)/(k_2 - u_2),$$

$$n_3 = \lambda(v_1 - u_1)/(k_1 - u_1) = \lambda(v_2 - u_2)/(k_2 - u_2).$$

Thus $n_1 = n_3$, and so

$$(v_1 - u_1 - 1)(k_1 - u_1) = (v_1 - u_1)(k_1 - u_1 - 1).$$

From this we obtain $-k_1 + u_1 = -v_1 + u_1$, and so $v_1 = k_1$. Applying Lemma 4.1, we obtain $\mathbf{k} = \mathbf{v}$, and the proof is complete. ∎

We remark that the block multiset of a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $\mathbf{k} = \mathbf{v}$ consists of the complete point-set repeated λ times. A t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $\mathbf{k} = (k-1, 1)$ (or $(1, k-1)$) and $k > t > 0$ is more interesting. In such a design, the point set is the disjoint union of a v_1-set V_1 and a v_2-set V_2, a block consists of a $(k-1)$-subset of V_1, together with a "label" from V_2. Restricting the blocks to V_1, we get a t-$(v_1, k-1, \lambda)$ design, and the labels from V_2 define a partition of this t-design into $(t-1)$-$(v_1, k-1, \lambda)$ designs.

We additionally remark that the preceding theorem says nothing about the interesting case of t-$(\mathbf{v}, \mathbf{k}, \lambda)$ designs with block size $k = t+1$. Many interesting examples of such designs, with small t and k, are studied in [1].

5 Intersection numbers of order r for generalised t-designs

Mendelsohn [5] introduced the concept of intersection numbers of order r for a t-design, with respect to a block of that design, and showed that these intersection numbers satisfy a certain system of integer linear equations. The concept of intersection numbers of order r, and Mendelsohn's equations, have since been generalised and applied to block designs which are not necessarily t-designs and to graphs (see [8, 2, 7]).

In this section we provide another generalisation of intersection numbers of order r and the equations they satisfy, this time in a way appropriate for generalised t-designs.

Let $D = (\mathbf{V}, \mathcal{B})$ be a generalised t-design, and let $S \subseteq V(D)$ with $s := |S| \geq t$ and $\mathbf{s} := [S]_{\mathbf{V}}$. Further, let $\mathbf{i}$ be a tuple of non-negative integers with $\mathbf{i} \leq \mathbf{s}$. Then for r a positive integer, the $\mathbf{i}$-th *intersection number of order* r, with respect to S, denoted $n_{\mathbf{i}}^{(r)}(S)$ (or $n_{\mathbf{i}}^{(r)}(D, S)$), is defined to be the number of collections of r blocks of D whose intersection intersects S in a set of $\mathbf{V}$-type $\mathbf{i}$. In particular, $n_{\mathbf{i}}^{(1)}(S)$ is the number of blocks of D (including repeats) intersecting S in a set of $\mathbf{V}$-type $\mathbf{i}$.

We need one further piece of notation before stating a theorem. Given any two m-vectors $\mathbf{r} = (r_1, \ldots, r_m)$, $\mathbf{s} = (s_1, \ldots, s_m)$ of non-negative integers, we write

$$\binom{\mathbf{r}}{\mathbf{s}} := \prod_{i=1}^{m} \binom{r_i}{s_i}.$$

Thus, if R is a finite set, $\mathbf{R}$ an ordered partition of R, and $\mathbf{r} = [R]_{\mathbf{R}}$, then $\binom{\mathbf{r}}{\mathbf{s}}$ is the number of subsets S of R with $[S]_{\mathbf{R}} = \mathbf{s}$.

Theorem 5.1 *Let $D = (\mathbf{V}, \mathcal{B})$ be a generalised t-design, and let all types be with respect to $\mathbf{V}$. Let $S \subseteq V(D)$ with $s := |S| \geq t$ and let r be a positive integer. Then for each tuple $\mathbf{j}$ of non-negative integers such that $|\mathbf{j}| \leq t$ and $\mathbf{j} \leq \mathbf{s} := [S]$, we have:*

$$\sum_{\mathbf{j} \leq \mathbf{i} \leq \mathbf{s}} \binom{\mathbf{i}}{\mathbf{j}} n_{\mathbf{i}}^{(r)}(S) = \binom{\mathbf{s}}{\mathbf{j}} \binom{\lambda_{\mathbf{j}}}{r}$$

(where $\lambda_{\mathbf{j}}$ is the number of blocks containing a point-subset of type $\mathbf{j}$).

Proof. Let $\mathbf{j}$ be a tuple of non-negative integers with $|\mathbf{j}| \leq t$ and $\mathbf{j} \leq \mathbf{s}$, and count in two ways the number $N_{\mathbf{j}}$ of ordered pairs (R, J) such that R is a collection (multiset) of r blocks of D and J is a subset, of type $\mathbf{j}$, of both S and $\cap_{B \in R} B$.

Now each subset J of S with $[J] = \mathbf{j}$ contributes exactly $\binom{\lambda_{\mathbf{j}}}{r}$ pairs of the form $(*, J)$ to $N_{\mathbf{j}}$, and so

$$N_{\mathbf{j}} = \sum_{J \subseteq S, [J] = \mathbf{j}} \binom{\lambda_{\mathbf{j}}}{r} = \binom{\mathbf{s}}{\mathbf{j}} \binom{\lambda_{\mathbf{j}}}{r}.$$

On the other hand, each collection R of r blocks of D contributes exactly

$$\binom{[\bigcap_{B \in R} B \cap S]}{\mathbf{j}}$$

pairs of the form $(R, *)$ to $N_{\mathbf{j}}$, and so

$$N_{\mathbf{j}} = \sum_{R \subseteq \mathcal{B}, |R| = r} \binom{[\bigcap_{B \in R} B \cap S]}{\mathbf{j}} = \sum_{\mathbf{j} \leq \mathbf{i} \leq \mathbf{s}} \binom{\mathbf{i}}{\mathbf{j}} n_{\mathbf{i}}^{(r)}(S).$$

Hence the result. ■

We remark that this result is proved for the case when D is a t-design and S is a block in [5], and in general for t-designs in [8]. A different generalisation to graphs and general block designs is given in [7] (from which we have adapted our proof of Theorem 5.1).

Example 4 Let $D = (\mathbf{V}, \mathcal{B})$ be any 2-$((6, 10), (3, 4), (\lambda_{(0,2)} = 4, \lambda_{(1,1)} = 6, \lambda_{(2,0)} = 6))$ design. We apply our generalisation of Mendelsohn's equations in the case $r = 1$ to obtain some information about the $\mathbf{V}$-types of intersections of blocks of D.

Let B be a block of D, $\mathbf{k} := [B]_{\mathbf{V}} = (3, 4)$, and let $n_{\mathbf{i}} := n_{\mathbf{i}}^{(1)}(B)$ for the (twenty) non-negative integer tuples $\mathbf{i} \leq \mathbf{k}$. Then for each $\mathbf{j} \in \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)\}$, we have

$$\sum_{\mathbf{j} \leq \mathbf{i} \leq \mathbf{k}} \binom{\mathbf{i}}{\mathbf{j}} n_{\mathbf{i}} = \binom{\mathbf{k}}{\mathbf{j}} \lambda_{\mathbf{j}},$$

as well as the inequalities

$$n_{\mathbf{i}} \geq 0 \;\text{ if }\; \mathbf{i} < \mathbf{k}, \quad \text{and} \quad n_{\mathbf{k}} \geq 1.$$

Now these conditions, possibly together with other linear constraints involving the $n_{\mathbf{i}}$, can be studied using exact linear or integer programming methods. Here we make use of the exact linear programming package `simplex` in the computer algebra system `Maple` [4].

For example, after adding to the conditions above the two inequalities $n_{(1,4)} \geq 1$ and $n_{(2,3)} \geq 1$, the `simplex` package function `feasible` informs us that there are no solutions with rational $n_{\mathbf{i}}$ (and hence none with integer $n_{\mathbf{i}}$). This tells us that in D, there can be no blocks B, X, Y, with $B \cap X$ having $\mathbf{V}$-type $(1, 4)$ and $B \cap Y$ having $\mathbf{V}$-type $(2, 3)$.

Similarly, we find no rational solutions when we require $n_{\mathbf{k}} \geq 2$, which shows that D must be simple; that is, D has no repeated blocks.

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