Pairwise balanced designs

One of the most important breakthroughs in design theory was made by Richard Wilson in the early 1970s. He showed that the trivial necessary conditions for the existence of various kinds of designs are asymptotically sufficient. For example, given a positive integer $k > 1$, the necessary conditions for the existence of a 2-$(v,k,1)$ design are that $k-1$ divides $v-1$ and $k(k-1)$ divides $v(v-1)$; it follows from Wilson's theorem that, given k, a design exists for all but finitely many v satisfying these conditions.

For the purpose of describing Wilson's existence theory, we define a *pairwise balanced design*, or *PBD*, to be a set P of points with a collection B of subsets (called blocks) such that any two points lie in a unique block. If K is the set of cardinalities of blocks, we refer to such a design as a PBD(K). Thus, if $K = \{k\}$, a PBD(K) with v points is a Steiner system $S(2,k,v)$.

We allow the possibility that $0,1 \in K$. Indeed, adding blocks with cardinality at most 1 does not affect the defining property of a PBD. Apart from this, a PBD is the same thing as a *linear space*.

Given a set K of non-negative integers, we let $\mathbb{B}(K)$ denote the set of numbers v for which a PBD(K) with v points exists. For example, $\mathbb{B}(\{3\})$ is the set of orders of *Steiner triple systems* (including degenerate ones on 0, 1 or 3 points).

We observe that $0,1 \in \mathbb{B}(K)$. Furthermore, $\mathbb{B}$ is a *closure operator* on the set of subsets of $\mathbb{N}$; that is,

- $K \subseteq \mathbb{B}(K)$;
- $K_1 \subseteq K_2$ implies $\mathbb{B}(K_1) \subseteq \mathbb{B}(K_2)$;
- $\mathbb{B}(\mathbb{B}(K)) = \mathbb{B}(K)$.

(For the first point, note that a set of size k with a single block containing all points is a PBD(K) if $k \in K$. The second point is obvious. The first two points immediately give $\mathbb{B}(K) \subseteq \mathbb{B}(\mathbb{B}(K))$. For the reverse implication, if $v \in \mathbb{B}(\mathbb{B}(K))$, then there is a PBD on v points whose block cardinalities are sizes of PBD(K)s; replace each block by some PBD(K) to obtain a PBD(K) on v points.)

Let K be a set of non-negative integers. Define two numbers $\alpha(K)$ and $\beta(K)$ by

$$\begin{aligned}\alpha(K) &= \gcd\{k-1 : k \in K\},\\ \beta(K) &= \gcd\{k(k-1) : k \in K\}.\end{aligned}$$

Proposition 1 *If* $v \in \mathbb{B}(K)$ *with* $v > 0$*, then* $\alpha(K)$ *divides* $v-1$*, and* $\beta(K)$ *divides* $v(v-1)$.

The result generalises the well-known fact that the order of a Steiner triple system is congruent to 1 or 3 mod 6, and the proof is almost identical.

Wilson's main theorem is a near converse:

Theorem 2 *Let* K *be a set of non-negative integers. Then all but finitely many integers* v *satisfying the conditions that* $\alpha(K)$ *divides* $v-1$ *and* $\beta(K)$ *divides* $v(v-1)$ *belong to* $\mathbb{B}(K)$.

This powerful theorem has many consequences. For example, it shows that there are only countably many PBD-closed sets. It shows that the necessary conditions that $k-1$ divides $v-1$ and $k(k-1)$ divides $v(v-1)$ are sufficient for the existence of a Steiner system $S(2,k,v)$ except for finitely many values of v, as claimed in the introduction.

There are consequences for other design construction problems as well:

Proposition 3 *Each of the following sets is PBD-closed:*

(a) The set of v *for which a* 2-(v,k,λ) *design exists, for any given* k *and* λ*. (Repeated blocks are permitted here.)*

(b) The set of n *for which* s *mutually orthogonal Latin squares of order* n *with a common transversal exist, for any given* s*.*

(c) The set of n *for which the edges of the complete graph* K_n *can be partitioned into copies of a given graph* G*.*

(d) The set of r *for which a resolvable* 2-$(v,k,1)$ *design with* r *parallel classes exists (given* k*).*

(e) The set of sides of Room squares.

The proof of part (a) is simple but illustrative. Suppose that $v \in \mathbb{B}(K)$, and suppose that a 2-(l,k,λ) design exists for each $l \in K$. Take a PBD(K) on a set of v points. Each block of this PBD carries a 2-design with block size k; we just take all the k-sets occurring as blocks in these designs. Now any two distinct points lie in a unique block of the PBD, and hence in exactly λ blocks of the constructed design, as required.

For example, the theorem asserts that $\mathbb{B}(\{4,7\})$ contains all but finitely many integers congruent to 1 mod 3; further analysis shows that the only exceptions are 10 and 19. Since there exist 2-$(4,4,2)$ and 2-$(7,4,2)$ designs, we conclude that there exist 2-$(v,4,2)$ designs for all v congruent to 1 mod 3 except possibly $v = 10$ and $v = 19$. The general existence question is thus reduced to just two cases (in both of which the designs exist, as it happens).

In the second example, the PBD-closed set K contains all prime powers $n > s+1$, and so $\alpha(K) = \beta(K) = 1$. We conclude that there is a number $n_0(s)$ such that s MOLS of order n exist for all $n \geq n_0(s)$.

Another way of describing a PBD-closed set K is by giving its *base* X, the set of all elements $x \in K$ such that $x \notin \mathbb{B}(K \setminus \{x\})$. It can be shown that this set is finite, and is the unique minimal set satisfying $\mathbb{B}(X) = K$. For example, the base for the set of orders of Steiner triple systems is $\{3\}$.

It is not possible to give more than a brief outline of the proof of Wilson's theorem here. The proof, though elaborate, is in the spirit of many constructions of designs, and uses both direct and recursive construction methods.

The direct methods show that PBDs with block sizes in K exist for all sufficiently large prime powers q satisfying the divisibility conditions. These constructions use cyclotomy to construct a family of subsets of GF(q), with sizes from K, such that any non-zero element of GF(q) is uniquely expressible as a difference of two elements from the same set.

The recursive constructions then build new PBDs from old, showing that a PBD of any sufficiently large size satisfying the necessary conditions will exist. Many of the recursive constructions are conveniently expressed in terms of group divisible designs (GDDs). We refer to Wilson's papers for details.

There is an exposition of the proof in Wilson's survey article [2]. The theory is developed in a series of papers [1].

References

[1] R. M. Wilson, An existence theory for pairwise balanced designs: I, Composition theorems and morphisms, *J. Combinatorial Theory* (A) **13** (1972), 220–245; II, The structure of PBD-closed sets and the existence conjectures, *ibid.* **13** (1972), 246–273; III, A proof of the existence conjectures, *ibid.* **18** (1975), 71–79.

[2] R. M. Wilson, Construction and uses of pairwise balanced designs, *Mathematical Centre Tracts* **55**, Mathematisch Centrum, Amsterdam, 1974.

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