Combinatorial and statistical design

1 Combinatorial design

To readers of books such as those of Hughes and Piper [3] or Beth, Jungnickel and Lenz [1], the term "design" means one or other of the following concepts:

- a set P of *points* and a set B of *blocks*, with an *incidence relation* between them (a subset I of $P \times B$), satisfying certain conditions; or
- a set P of *points* and a set or family B of subsets of P called *blocks*, satisfying certain conditions.

Typical conditions are given below. A structure of the first type is called an*incidence structure*.

The two versions amount to the same thing. Given a design of the first type (specified by an incidence relation), we associate with each $b \in B$ the subset

$$X(b) = \{p \in P : (p,b) \in I\}$$

of P, to obtain an indexed family

$$B^* = (X(b) : b \in B)$$

of subsets of P. Note that, in general, we have a family rather than a set of subsets; there may be *repeated blocks* (that is, $X(b) = X(b')$ for some $b \neq b'$). Conversely, if $(X_b : b \in B)$ is a family of subsets of P indexed by B, we take

$$I = \{(p,b) : p \in X_b)\}$$

to obtain an incidence structure.

The language used in design theory reflects this dual interpretation. Thus, we often say that a point p is *incident* with a block b if (in the second interpretation) p lies in the subset of P corresponding to b.

Note that sometimes the word "design" is restricted to structures with no repeated blocks. (This is the convention in [3], for example.)

Typical conditions that are imposed include:

(a) any block is incident with exactly k points (and it is often further assumed that $1 < k < v$, where $v = |P|$);

(b) any t points are incident with exactly λ blocks, where t and λ are parameters with $\lambda > 0$.

A structure satisfying (a) and (b) is called a t-(v,k,λ) *design*, or a t-*design* for short. The unadorned word "design" is used by several authors to mean "2-design".

A structure satisfying (b) is called a t-*wise balanced design*. For $t = 2$, it is a *pairwise balanced design*: for $t = 2$ and $\lambda = 1$ it is a *linear space*. (Often it is forbidden that a block is incident with fewer than t points; such blocks can be removed without affecting (b).) More generally, a *partial linear space* is an incidence structure with the property that two points are incident with at most one block.

A structure satisfying (a) and having no repeated blocks is called a k-*uniform hypergraph*. However, typical concerns of hypergraph theory are different from those of design theory.

For example, let $P = \{1,\ldots,7\}$ and

$$B = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\},$$

It is readily checked that this is a 2-$(7,3,1)$ design, or linear space. This design is the *projective plane* of order 2.

The *dual* of an incidence structure $(P,B,I)$is the incidence structure (B,P,I^*), where $I^* = \{(b,p) : (p,b) \in I\}$. Note that the dual of a partial linear space is also a partial linear space. We mention here an important result from design theory:

Theorem 1 *Let (P,B,I) be a 2-(v,k,λ) design with $1 < k < v-1$ and $\lambda > 0$. Then $|B| \geq |P|$, with equality if and only if the dual is also a 2-design.*

The inequality in this theorem is known as *Fisher's Inequality*. A design attaining the bound is variously called *square*, *symmetric*, or (in Dembowski [2]) *projective*. The example given above is a square design; it can be checked that it is isomorphic to its dual.

There is yet another representation of an incidence structure that is important to us. Let (P,B,I) be an incidence structure. The *incidence graph* or *Levi graph* of the structure is the bipartite graph with vertex set $P \cup B$, in which p is adjacent to b whenever $(p,b) \in I$.

Conversely, suppose that we have a bipartite graph with bipartite sets P and B. Taking I to be the set of edges (each edge ordered so that its first vertex is in P), we obtain an incidence structure.

The Levi graph of the dual is the same as that of the original structure; only the labels *point* and *block* are interchanged.

2 Experimental design

In a typical agricultural or medical experiment, we have a discrete set Ω of *plots* or experimental units, to which certain treatments will be applied. An *experimental design* is a function F from Ω to the set T of treatments, where $F(\omega)$ is the treatment to be applied to the plot ω. The design F gives rise to a *factor* or partition of Ω, where two plots are in the same part if and only if they receive the same treatment. Thus, the choice of experimental design can be divided into two stages: the choice of a factor, and the allocation of treatments to parts (which may involve the choice of a random bijection).

Two features of the set-up complicate this simple picture.

First, the set T may be structured. For example:

(a) One of the treatments may be 'no treatment' or 'placebo'. This may require special consideration.

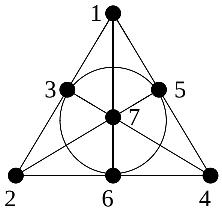
(b) The treatments may consist of several different levels of each of a number of fertilisers, say. In this case we have a partition of T, each part of which is ordered or enumerated. The partition of T induces a partition of Ω coarser than the design factor.

(c) T may consist of, say, combinations of a fertiliser and a watering regime. In this case we have two partitions of T.

We ignore this for the moment and consider the second complication, which arises even when the treatment set is unstructured. The set of plots may itself be structured. In a medical trial, patients are either male or female, and are recruited in different hospitals where practices may vary. A fertiliser trial may be conducted on several farms in different parts of the country, where soil conditions are quite different. In general, there may be 'nuisance factors' or partitions of Ω which have an effect on the experiment but are outside the control of the experimentalist. The statistician has to cope not only with random variation but also with systematic variation resulting from such factors.

Specifically, suppose that seven brands of fertiliser are to be tested. Twenty-one fields are available, three on each of seven farms. We want to apply each fertiliser to three fields on different farms. For reasons to be discussed later, the best design is that shown in Figure 1.

We recognise here the projective plane of order 2 from the last section. However, the viewpoint is different: the basic units are the 21 plots, and there are two



1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

Figure 1: An experimental design

distinguished partitions, one corresponding to the seven farms, and the other (the *design factor*) to the seven types of fertiliser.

Statisticians would regard the farms (or, more precisely, the parts of the corresponding partition of Ω) as *blocks*. A set with two partitions is called a *block design*.

Our assumption that no fertiliser is used more than once on each farm says that the meet of the two partitions (in the lattice of all partitions of Ω) is equality, the partition whose parts are singletons. A block design with this property is called *binary*.

Given a binary block design, let P and B denote the sets of treatments and blocks respectively. Define an incidence relation $I \subseteq P \times B$ by the rule that $(p,b) \in I$ if the intersection of part p of P and part b of B is non-empty (informally, treatment p occurs on some plot in block b).

Conversely, given an incidence structure (P,B,I), we obtain a binary block design as follows; take the set Ω of plots to be I; the part p of the partition P consists of all pairs in I with first component p; the part b of the partition B consists of all pairs with second component b. (We can also think of Ω as the set of edges of the Levi graph; the two partitions correspond to the vertices in the two bipartite blocks.)

The block design given above has two further properties:

- There are seven treatments (types of fertiliser), but each block (farm) can only accommodate three, so not every treatment appears in every block. A block design with this property is called *incomplete*. (It is better to say that it is an *incomplete-block design*.)

- It is *balanced*, that is, each pair of treatments occurs together the same number of times (in this case, once) in each block. Intuitively, this is a good feature enabling fair comparison between the treatments. This can be made more precise using this notion of *efficiency*.

Clearly a binary balanced incomplete-block design (for short, BIBD or BIB) translates as above to a 2-(v,k,λ) design with $k < v$.

3 Other types of design

The statistician's notion of design is thus more general than the combinatorialist's: incidence structures correspond to binary block designs. In this section we look at two different types of experimental design.

3.1 Latin squares

Suppose that five judges taste five different wines; the tests are arranged in five rounds, so that each judge tastes each wine once. A plot in this experiment consists of one judge tasting one wine; there are 25 plots. The set Ω of plots carries three partitions, each with five parts of size 5:

- a partition corresponding to the five judges;
- a partition corresponding to the five wines;
- a partition corresponding to the five tasting rounds.

Here the wines are the treatments we want to compare, while the judges and rounds are the nuisance factors or blocks. Taking any two of the three partitions, we get a complete-block design (whose Levi graph is complete bipartite); the structure of the design is not captured by the corresponding incidence structures.

We can represent the design as a square array in which each cell contains one of the numbers $1,\ldots,5$, each symbol occurring once in each row or column, as in Figure 2. Such an array is called a *Latin square*; its cells correspond to the plots, and the three partitions are given by rows, columns, and symbols.

It is possible to represent a Latin square as an incidence structure in a different way: we take the point set to be the set Ω of cells, and the blocks to be the parts of the three different partitions. The result is a special type of partial linear space called a *net*. It has the property that, if b is a block and p a point not incident with b, then exactly one block is incident with p and disjoint from p. (This is a version of Playfair's Axiom, a form of Euclid's parallel postulate.)

1	2	3	4	5
2	3	1	5	4
3	4	5	2	1
4	5	2	1	3
5	1	4	3	2

Figure 2: A Latin square

3.2 Youden "squares"

A symmetric balanced incomplete-block design (SBIBD), or square 2-design, can (like any incidence structure) be represented by its incidence graph, a bipartite graph with parts X_1 and X_2. The graph has the properties

- $|X_1| = |X_2| = v$;
- for $\{i, j\} = \{1, 2\}$, any point in X_i has exactly k neighbours in X_j;
- for $\{i, j\} = \{1, 2\}$, any two points in X_i have exactly λ neighbours in X_j.

Any regular bipartite graph has a 1-*factorisation*, a partition of the edge set into k parts or 1-factors of v edges each, where the edges of each 1-factor partition the vertices. The structure given by a square 2-design and a 1-factorisation of its incidence graph is called a *Youden square*. It can be represented in various ways other than as an edge-coloured bipartite graph; for example:

- As a set with three partitions: the set C is the set of edges of the graph (or flags in the design); there is a partition $\mathcal{A}$ into k sets of size v given by the 1-factorisation; and there are two partitions $\mathcal{B}_1$ and $\mathcal{B}_2$ into v sets of size k corresponding to the sets X_1 and X_2, where parts in $\mathcal{B}_i$ are labelled by vertices in X_i, the part labelled p consisting of all edges incident with p. Note that the partitions $\mathcal{A}$ and $\mathcal{B}_i$ are *orthogonal* (in the sense that a part of $\mathcal{A}$ and a part of $\mathcal{B}_i$ meet in one point). Also, parts labelled by $p_1 \in X_1$ and $p_2 \in X_2$ meet in at most one point, the intersection being non-empty if and only if p_1 and p_2 are incident. So the original design (as incidence structure) and the 1-factorisation of its incidence graph can be recovered from the set of partitions.

- As a square array: number the 1-factors from 1 to k and the points of X_1 and X_2 from 1 to v. Then take the $v \times v$ matrix whose (i, j) entry is equal to l if the ith point of X_1 and the jth point of X_2 are incident and the edge joining them belongs to the lth 1-factor, and is blank otherwise. (Replacing all non-blank entries by 1 and blanks by 0 gives the *incidence matrix* of the design.) This is the representation used by Fisher in presenting Youden's concept, and is probably the reason why they are called "squares", whereas the following representation would suggest "rectangles".

- As a Latin rectangle: with the above numbering, take the $k \times v$ array whose (i, j) entry is l if the lth point of X_2 is joined to the jth point of X_1 by an edge of the ith 1-factor. This is the representation used by Youden, and is the one most commonly used in view of its compactness, although it obscures the symmetry between X_1 and X_2.

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory* (2 volumes), Cambridge University Press, Cambridge, 1999.

[2] P. Dembowski, *Finite Geometries*, Springer, Berlin, 1968.

[3] D. R. Hughes and F. C. Piper, *Design Theory*, Cambridge University Press, Cambridge, 1985.

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