Cayley graphs and coset diagrams

1 Introduction

Let G be a finite group, and X a subset of G. The *Cayley graph* of G with respect to X, written $\mathrm{Cay}(G,X)$ has two different definitions in the literature. The vertex set of this graph is the group G. In one definition, there is an arc from g to xg for all $g \in G$ and $x \in X$; in the other definition, for the same pairs (g,x), there is an arc from g to gx.

Cayley graphs are generalised by *coset graphs*. For these we take a subgroup H of G as part of the data. Now the vertices of the graph may be either the left cosets or the right cosets of H in G; and an arc may join the coset containing g to the coset containing xg, or to the coset containing gx. Thus, there are four possible types of coset graphs.

The two types of Cayley graph are in a certain sense equivalent, while the four types of coset graph fall into two distinct types: one type encompasses all vertex-transitive graphs, while the other is seldom vertex-transitive but has more specialised uses in the theory of regular maps. The purpose of this essay is to explain where the definitions come from and what purposes they serve.

2 Group actions

An *action* of a group G on a set Ω is a homomorphism from G to the symmetric group on Ω. This simply means that to each group element g is associated a permutation π_G of Ω, and composition of group elements corresponds to composition of permutations: $\pi_{g_1g_2} = \pi_{g_1} \circ \pi_{g_2}$, where $\circ$ denotes composition.

However, there are two different conventions for composing permutations. These arise from different ways of representing a permutation π: the image of a point x under the permutation π may be denoted by $\pi(x)$ or by $x\pi$ (or x^{π}). In the first case, we say the permutation acts *on the left*, in the second case *on the right*.

If π_1 and π_2 are permutations, then $\pi_1 \circ \pi_2$ may mean "apply first π_1, then π_2", or it may mean "apply first π_2, then π_1". Now the definition we choose of $\circ$ is closely connected with the way we choose to write permutations. If permutations act on the left, then we would like to have

$$(\pi_1 \circ \pi_2)(x) = \pi_1(\pi_2(x)),$$

that is, "first π_2, then π_1"; while if they act on the right, we would prefer

$$x(\pi_1 \circ \pi_2) = (x\pi_1)\pi_2 \qquad \text{or} \qquad x^{\pi_1 \circ \pi_2} = (x^{\pi_1})^{\pi_2}.$$

When there is a choice, I will use the right action, since the definition of composition under the right action is so much more natural. I will simplify the notation by writing simply xg for the image of x under the permutation π_g. The rule for an action becomes simply $x(gh) = (xg)h$.

Now suppose that G acts (on the right) on a set Ω. The action is *transitive* if we can move from any point of Ω to any other by some permutation induced by G; that is, for any $\alpha, \beta \in \Omega$ there is an element $g \in G$ such that $\alpha g = \beta$.

Two actions of G, on sets Ω_1 and Ω_2, are said to be *isomorphic* if there is a bijection $\phi : \Omega_1 \to \Omega_2$ (which I shall also write on the right) such that $(\alpha\phi)g = (\alpha g)\phi$ for all $\alpha \in \Omega_1$ and $g \in G$. In other words, up to renaming the points of the set, the actions are identical.

Now let H be a subgroup of G. The *right cosets* of H are the sets $Hx = \{hx : h \in H\}$, and the *left cosets* are the sets $xH = \{xh : h \in H\}$, as x runs through G. Two cosets of the same type are either equal or disjoint; two elements x_1, x_2 lie in the same right coset if and only if $x_1 x_2^{-1} \in H$, and lie in the same left coset if and only if $x_2^{-1} x_1 \in H$. So the cosets of each type form a partition of G. The subgroup H is *normal* in G if and only if the partitions into left and right cosets coincide. It is worth noting that there is a natural bijection between the right cosets and the left cosets: to the right coset Hx corresponds the left coset

$$x^{-1}H = \{g^{-1} : g \in Hx\}.$$

The sets of left and right cosets of H are denoted by G/H and $H\backslash G$ respectively. (The position of G relative to H in the notation tells where the coset representative should be put.)

Now there is an action ρ of G on the set of right cosets of H by the rule

$$(Hx)\rho_g = H(xg).$$

This means that the permutation ρ_g corresponding to g maps the coset Hx to the coset $H(xg)$. (One must show that the image is independent of the choice of coset representative x, so that the map is well-defined; that the map is a permutation; and that the condition for an action holds. All of this is straightforward.) We can write this in our simpler notation as $(Hx)g = H(xg)$ without too much confusion. Similarly, there is an action λ on the set of left cosets of H given by

$$(xH)\lambda_g = (g^{-1}x)H.$$

(The inverse is required to make the action a homomorphism.) These two actions are isomorphic: the bijection in the preceding paragraph satisfies the conditions for an isomorphism of actions. We will concentrate on the action on right cosets. Clearly this is transitive.

Conversely, every transitive action is isomorphic to an action on the right cosets of a subgroup. If G acts on Ω, and $\alpha \in \Omega$, the *stabiliser* of α is the subset

$$G_\alpha = \{g \in G : \alpha g = \alpha\}.$$

It is a subgroup of G. If G acts transitively on Ω, then for each $\beta \in \Omega$, the set

$$X(\alpha,\beta) = \{g \in G : \alpha g = \beta\}$$

is a right coset of G_α, and the map $\beta \to X(\alpha,\beta)$ is an isomorphism from the given action to the action on right cosets of G_α.

3 Graphs and permutations

We will consider directed graphs, whose arcs are ordered pairs of vertices. An undirected graph will be a directed graph with the property that if (v,w) is an arc then so is (w,v); if this holds we speak of the edge $\{v,w\}$.

A function f on a set Ω can be regarded as the set of all ordered pairs $(\alpha, f(\alpha))$ for $\alpha \in \Omega$. We can regard these pairs as the arcs of a digraph, the *functional digraph* $\Phi(f)$. A digraph is a functional digraph if and only if every vertex has exactly one arc leaving it. The function f is a permutation if and only if every vertex of $\Phi(f)$ has exactly one arc entering it. If so, then $\Phi(f^{-1})$ is obtained simply by reversing all the arcs.

Proposition 1 *If f and π are permutations of Ω, then π is an automorphism of the functional digraph $\Phi(f)$ if and only if π and f commute.*

Proof

$$(\alpha\pi, f(\alpha)\pi) \in \Phi(f) \Leftrightarrow f(\alpha)\pi = f(\alpha\pi).$$

Note the asymmetry: we have written f on the left and π on the right. This makes the commuting condition appear more natural!

See [1] for more information on graph automorphisms.

4 Cayley graphs

An action of G on Ω is said to be *regular* if it is transitive and the stabiliser of a point consists of the identity element of G only. Now the cosets (left or right) of the identity subgroup are the singleton subsets of G, which can be naturally identified with the elements of G. So any regular action of G is isomorphic to the "action of G on itself by right multiplication", given by $x\pi_g = xg$ for $x, g \in G$. The same result would be true if we used left multiplication.

The *Cayley graph* of G with *connection set* $X \subseteq G$ is defined to be the directed graph whose arc set is the union of the arc sets of the functional digraphs $\Phi(\lambda_x)$, for $x \in X$. In other words, (g, xg) is an arc for all $g \in G$ and $x \in X$. We denote this graph by $\mathrm{Cay}(G, X)$.

Proposition 2 *A digraph Γ with vertex set G admits G (acting by right multiplication) as a group of automorphisms if and only if Γ is a Cayley graph* $\mathrm{Cay}(G,X)$ *for some $X \subseteq G$.*

Proof In one direction we use the observation that the associative law for a group G is a "commutative law" for the left and right multiplications:

$$(x\lambda_g)\rho_h = (gx)\rho_h = gxh = (xh)\lambda_g = (x\rho_h)\lambda_g.$$

So if Γ is a Cayley graph, then right multiplication preserves the arc sets of all the functional digraphs $\Phi(\lambda_x)$ for $x \in X$, and hence the arc set of Γ.

Conversely, suppose that Γ admits the right action of G. Let

$$X = \{x \in G : (1, x) \text{ is an arc of } \Gamma\}.$$

Then applying g on the right we see that (g, xg) is an arc, for all $x \in X$ and $g \in G$. Conversely, if (g, h) is an arc, then $(1, hg^{-1})$ is an arc, so $hg^{-1} \in X$. Thus $\Gamma = \mathrm{Cay}(G, X)$.

Proposition 3 *The Cayley digraph* $\mathrm{Cay}(G,X)$ *is loopless if and only if* $1 \notin X$*; it is undirected if and only if $X = X^{-1}$; and it is connected if and only if X generates G. The group G acts vertex-transitively on* $\mathrm{Cay}(G,X)$.

If "left" and "right" are reversed throughout this section (so that a Cayley graph is a union of functional digraphs for the right action, and automorphisms act on the left), then an equivalent theory is obtained. The literature is divided on this point!

5 Vertex-transitive graphs

Now there are two ways of generalising Cayley graphs:

- We may take the definition as a union of functional digraphs of permutations, and use arbitrary permutations;
- We may regard vertex-transitivity as the important property and impose that.

In this section I will consider the second approach.

First a small digression. If H is a subgroup of G, then an H-H *double coset* is a subset of G of the form

$$HxH = \{h_1xh_2 : h_1, h_2 \in H\}.$$

As with right and left cosets, it holds that G is a disjoint union of double cosets. However, the double cosets do not all have the same size: we have

$$|HxH| = \frac{|H|^2}{|H \cap x^{-1}Hx|}.$$

For it is easy to show that the denominator is the number of ways of writing an element in the form h_1xh_2 for $h_1, h_2 \in H$. The set of double cosets is written $H\backslash G/H$. Double cosets of different subgroups H and K can also be defined but we do not require this.

Now suppose that Γ is a graph with vertex set Ω, and G a group of automorphisms of Γ acting vertex-transitively on Γ. As we saw earlier, the action of G on Ω is isomorphic to its action on the set of right cosets of a subgroup H of G (the stabiliser of a point α of Ω; so we can replace Ω by the set $H\backslash G$ of right cosets. How do we describe the arcs of Γ?

Following the proof in the preceding section, let

$$X = \{x \in G : (\alpha, \alpha x) \text{ is an arc of } \Gamma\}.$$

Then the following facts are easily seen:

- Hg_1 is joined to Hg_2 if and only if $g_2 = xg_1$ for some $x \in X$;
- X is a union of H-H double cosets;
- Γ is loopless if and only if $H \not\subseteq X$;

- Γ is undirected if and only if $X = X^{-1}$;
- Γ is connected if and only if X generates G;
- G acts arc-transitively on Γ if and only if X consists of just one double coset.

Conversely, if X is a union of H-H double cosets, and we define a digraph on $H\backslash G$ by the rule that Hg is joined to Hxg for all $x \in X$ and $g \in G$, then the resulting digraph, which we denote by $\Gamma(G,H,X)$, is vertex-transitive. All vertex-transitive digraphs (up to isomorphism) are thus produced by this construction:

Proposition 4 *Any vertex-transitive graph is isomorphic to a graph* $\Gamma(G,H,X)$, *where G is a group, H a subgroup of G, and X a union of H-H double cosets.*

6 Homomorphisms and Sabidussi's Theorem

A *homomorphism* from a digraph Γ_1 to a digraph Γ_2 is a map f from the vertex set of Γ_1 to that of Γ_2 with the property that, if (v,w) is an arc of Γ_1, then $(f(v), f(w))$ is an arc of Γ_2. (There is no requirement about the case when (v,w) is not an arc.)

Theorem 5 (Sabidussi [3]) *Every vertex-transitive graph is a homomorphic image of a Cayley graph.*

Proof We can represent our vertex-transitive digraph Γ as $\Gamma(G,H,X)$. Now let $\Gamma' = \mathrm{Cay}(G,X)$, and define $f(g) = Hg$. Any arc (g,xg) of Γ' is mapped to an arc (Hg,Hxg) of Γ.

For example, the Petersen graph is vertex-transitive but is not a Cayley graph, since its automorphism group has no transitive subgroup of order 10. However, the dodecahedron is a Cayley graph for the Frobenius group of order 20, and the map which identifies antipodal vertices induces a homomorphism from the dodecahedron to the Petersen graph.

7 Coset diagrams

Now we turn to the other possible generalisation of Cayley graphs. Given a set X of permutations of Ω, define $\Phi(X)$ to be the digraph whose arc sets are the unions of the arc sets of the functional digraphs $\Phi(f)$, for $f \in X$. We can regard the arcs of $\Phi(X)$ to be coloured by the elements of X, so that the arc $(\alpha, f(\alpha))$ has colour f. We call this graph a *coset diagram*.

There is not much to be said about this construction in general. It is useful in the theory of regular maps.

Suppose that M is a map embedded in a surface. A *dart* or *flag* of M is an ordered triple consisting of a mutually incident vertex, edge and face of M. If Ω is the set of darts of the map, then one can define three permutations of Ω:

- $a : (v,e,f) \mapsto (v',e,f)$, where v' is the other end of the edge e;
- $b : (v,e,f) \mapsto (v,e',f)$, where e' is the other edge incident with v and f;
- $c : (v,e,f) \mapsto (v,e,f')$, where f' is the face on the other side of the edge e.

Clearly a,b,c are involutions (that is, $a^2 = b^2 = c^2 = 1$). Moreover, ab maps a dart to the dart obtained by a rotation of the face f by one step; bc maps a dart to the dart obtained by a rotation of the edges at the vertex v by one step; and ac maps a dart to the dart obtained by reflecting in the midpoint of the edge e. Thus, we have

$$a^2 = b^2 = c^2 = (ab)^m = (bc)^n = (ac)^2 = 1,$$

where m and n are the least common multiples of the face sizes and vertex degrees of the map. The involutions a,b,c are fixed-point-free (acting on the set of darts), and the group they generate is transitive.

Conversely, given three fixed-point-free involutions a,b,c satisfying these relations and generating a transitive group G, there is a map on some surface encoded by the three permutations. The automorphism group of the map consists of permutations of the darts commuting with the permutations defining the map.

Now the permutations a,b,c are represented by coset diagrams for H in G, where H is the stabiliser of a dart. The advantage of a well-drawn coset diagram is that it makes it easy to check the relations satisfied by the three permutations. For example, if the diagram is symmetric about the vertical axis and ac is the reflection, then clearly $(ac)^2 = 1$.

For a fine example of a coset diagram, see the portrait of Graham Higman, by Norman Blamey, in the Mathematical Institute, Oxford. Higman was a pioneer of their use in this context [2]. A recent survey [4] is also recommended.

A coset diagram is not usually a vertex-transitive graph. In the special case where G acts regularly, a coset diagram for any set X of its elements is a Cayley graph $\mathrm{Cay}(G,X)$; and we have seen that it admits the vertex-transitive group G. A well-known result from permutation group theory asserts that if the centraliser of a transitive group G is also transitive, then G is regular; so the only coset diagrams which admit vertex-transitive groups of colour-preserving automorphisms are Cayley graphs.

Note that the rule for a coset diagram is that Hg is joined to Hgx for all $g \in G$, $x \in X$; compare the definition of the coset graph $\Gamma(G,H,X)$, where Hg is joined to Hxg.

8 Normal Cayley graphs

As we defined it, a Cayley graph for G is a graph on the vertex set G which admits the action of G on the right.

Proposition 6 *The following are equivalent for the Cayley graph* $\Gamma = \mathrm{Cay}(G,X)$*:*

(a) Γ *admits the action of G on the left;*

(b) the connection set X is a normal subset of G, that is, $g^{-1}Xg = X$ *for all* $g \in G$*;*

(c) the connection set X is a union of conjugacy classes in G.

Proof Clearly (b) and (c) are equivalent. If (a) holds, then left multiplication takes the arc $(1,x)$ to (g,gx); so $gxg^{-1} \in X$ for any $g \in G$, $x \in X$, whence X is a normal subset. The converse is shown in the same way.

Obviously, every Cayley graph for an abelian group is normal; but this is false for any non-abelian group.

The analogue of normal Cayley graphs for coset graphs has not been investigated. There is no "left action" of G on the set of right cosets of a non-normal subgroup H. However, conditions (b) and (c) make sense even in this more general context.

9 Other developments

If Γ is a Cayley graph for a group G, then certainly G is contained in the automorphism group of Γ. A question which received a lot of attention was: When is G the full automorphism group. The graph Γ is said to be a *graphical/digraphical* *regular representation* of G (GRR or DRR) if the full group of automorphisms of the graph or digraph Γ is just G.

Abelian groups with exponent greater than 2 never have GRRs. For if $\Gamma = \mathrm{Cay}(G,X)$ is undirected, then $X = X^{-1}$, and the map $g \mapsto g^{-1}$ is an automorphism of Γ not lying in G. Similar remarks hold for *generalised dicyclic groups*, those of the form $G = \langle A, t\rangle$, where $t^2 \in A$, $t^2 \neq 1$, and $t^{-1}at = a^{-1}$ for all $a \in A$. Hetzel and Godsil showed that apart from these and finitely many others (all determined explicitly), every group has a GRR. It is now thought that, apart from these two infinite families of exceptions, a random undirected Cayley graph for the group G is a GRR for G with high probability.

Other research areas include catalogues of vertex-transitive and Cayley graphs, quasi-Cayley graphs ("Cayley graphs for quasigroups"), and the very important area of infinite Cayley graphs of finite valency. Various generalisations of vertex-transitive graphs, such as graphs with constant neighbourhood, walk-regular graphs, and compact graphs, have also been studied. See [1] for some pointers to the literature.

References

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