Association schemes and permutation groups

This note discusses some properties of permutation groups related to association schemes. This material appears in [1, 4].

1 Permutation groups and coherent configurations

Let G be a permutation group on a set Ω of cardinality n. There is an induced action of G on the set Ω^2 of ordered pairs of elements of Ω, by the rule $(\alpha,\beta)g = (\alpha g, \beta g)$ for $\alpha,\beta \in \Omega$, $g \in G$. So Ω^2 is the disjoint union of G-orbits, called the *orbitals* of G on Ω. Each orbit O_i can be represented by an $n \times n$ matrix A_i, with rows and columns indexed by Ω, where

$$(A_i)_{\alpha,\beta} = \begin{cases} 1 & \text{if } (\alpha,\beta) \in O_i, \\ 0 & \text{otherwise.} \end{cases}$$

We call these the *orbital matrices* for G; their number is the *rank* of G.

The orbital matrices $A_1,\ldots,A_r$ satisfy

(CC1) $\sum_{i=1}^{r} A_i = J$, the all-1 matrix;

(CC2) there is a subset of $\{A_1,\ldots,A_r\}$ whose sum is the identity matrix I;

(CC3) for any i, there exists j such that $A_i^\top = A_j$;

(CC4) for any i, j, the product A_iA_j is a linear combination of $A_1,\ldots,A_r$.

A set of zero-one matrices satisfying (CC1)–(CC4) is called a *coherent configuration*. It is really a combinatorial object, since the conditions on the matrices can be translated into combinatorial conditions on the binary relations O_i. The coherent configuration formed by the orbital matrices of a permutation group G is the *orbital configuration* of G. Indeed, a coherent configuration is a partition of Ω^2 with some additional properties. So the coherent configurations on Ω inherit an order from the lattice of partitions of Ω^2. We say that the coherent configuration $\mathcal{A}$ is *finer than* $\mathcal{B}$ (or that $\mathcal{B}$ is *coarser than* $\mathcal{A}$) if each relation in $\mathcal{A}$ is a subset of some relation in $\mathcal{B}$.

The orbital configuration of a permutation group G is clearly the finest coherent configuration on which G acts as a group of *strong automorphisms* (preserving all the relations O_i).

2 Association schemes

An *association scheme* is a coherent configuration in which all the matrices are symmetric. (In other words, all the binary relations are symmetric.) It follows from the axioms that an association scheme contains the identity matrix (that is, the subset referred to in (CC2) contains just one element), and that the matrices commute with one another.

It is conventional to re-number the matrices as $A_0,\ldots,A_s$, where $s = r-1$, so that A_0 is the identity. Now a set $\{A_0,\ldots,A_s\}$ of zero-one matrices is an association scheme if the following conditions hold:

(AS1) $\sum_{i=0}^{s} A_i = J$, the all-1 matrix;

(AS2) $A_0 = I$;

(AS3) for any i, $A_i^\top = A_i$;

(AS4) for any i, j, the product A_iA_j is a linear combination of $A_0,\ldots,A_s$.

Association schemes are important in design theory because they may support *partially balanced incomplete block designs*. A block design D with point set Ω is partially balanced with respect to an association scheme $\mathcal{A}$ on Ω if, for any two points $\alpha,\beta \in \Omega$, the number of blocks containing α and β depends only on the *associate class* containing $\{\alpha,\beta\}$, that is, the index i such that $(A_i)_{\alpha,\beta} = 1$.

The following theorem is proved in [2]:

Theorem 1 *The supremum (in the partition lattice) of a set of association schemes on Ω is an association scheme.*

This has the following consequence. Suppose that a block design D is partially balanced with respect to some association scheme on Ω. Then there is a unique coarsest such scheme, namely the supremum of all association schemes with respect to which it is partially balanced. This scheme is called the *balancer* of D.

3 Association schemes and permutation groups

As we have seen, there is a coherent configuration associated with any permutation group, namely the orbital configuration. It is the unique finest configuration on which the group acts by strong automorphisms. Moreover, this configuration is *trivial* (consisting of the matrices I and $J-I$) if and only if the group is 2-transitive. (A permutation group G is 2-transitive if and only if it acts transitively on the set of ordered pairs of distinct points of Ω.)

For association schemes, the analogous statements are false. There is not always a unique finest G-invariant association scheme; and there are groups which are not 2-transitive or even 2-homogeneous but preserve only the trivial scheme. (A permutation group G is 2-homogeneous if and only if it acts transitively on the set of unordered pairs of distinct points of Ω.)

In the following discussion, we assume that the permutation group G is transitive on Ω.

We say that G is *AS-friendly* if there is a unique finest G-invariant association scheme. We say that G is *AS-free* if the only G-invariant association scheme is the trivial scheme $\{I, J-I\}$. Note that an AS-free group is AS-friendly, and that any 2-homogeneous group is AS-free.

We will see that these concepts are related to other well-studied properties, which we now define.

The group G is *primitive* if the only G-invariant partitions of Ω are the two trivial ones (the partition into singletons and the partition with a single part).

The group G is *generously transitive* if its orbital matrices are symmetric; it is *multiplicity-free* if its orbital matrices commute; and it is *stratifiable* if its symmetrised orbital matrices commute. (The *symmetrised* orbital matrices are obtained by replacing each distinct pair $A_i, A_i^\top$ by their sum $A_i + A_i^\top$.)

Each of these concepts clearly implies the next. (Use the fact that symmetric matrices commute if and only if their product is symmetric.) If G is stratifiable, then the symmetrised orbital matrices form an association scheme; and if G is generously transitive, the orbital matrices themselves form an association scheme. If G is stratifiable, we call the association scheme formed by the symmetrised orbital matrices the *orbital scheme* of G.

Clearly a permutation group G on Ω is generously transitive if and only if, for any distinct $\alpha, \beta \in \Omega$, there is an element of G which interchanges α and β. So this condition is clearly a strengthening of transitivity.

The three conditions can also be translated into conditions on the *permutation character* of G, the function counting fixed points of elements of G. Like any character, the permutation character can be expressed as a linear combination of irreducible characters with non-negative integer coefficients; those with positive coefficients are its *constituents*. Recall that the *type* of an irreducible character is "real" if it is the character of a real representation; "quaternionic" if it is real-valued but is not the character of a real representation; and is “complex” if it is not real-valued. The multiplicity of a quaternionic character in the permutation character is necessarily even. Now:

- *G* is generously transitive if and only if all irreducible constituents of the permutation character are real and have multiplicity 1;
- *G* is multiplicity-free if and only if all irreducible constituents of the permutation character have multiplicity 1 (this implies that they are real or complex);
- *G* is stratifiable if and only if all irreducible constituents of the permutation character have multiplicity 1, except possibly for quaternionic characters with multiplicity 2¿

In contrast, there seems no straightforward way to recognise whether *G* is AS-friendly, or AS-free.

These properties are related as follows:

Theorem 2 *The following implications hold between properties of a permutation group G:*

$$\begin{array}{ccccccc} 2\text{-}\textit{transitive} & \Rightarrow & 2\text{-}\textit{homogeneous} & \Rightarrow & \textit{AS-free} & \Rightarrow & \textit{primitive} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \textit{gen. trans.} & \Rightarrow & \textit{stratifiable} & \Rightarrow & \textit{AS-friendly} & \Rightarrow & \textit{transitive} \end{array}$$

None of these implications reverses, and no further implications hold.

Theorem 3 *Each property in the preceding theorem is closed upwards (that is, if G has a transitive subgroup H having some property, then G has that property).*

4 Classifications and remarks

Using the Classification of Finite Simple Groups, a complete list of the 2-transitive permutation groups is known. This is given in [3]. Moreover, all 2-homogeneous but not 2-transitive groups are also known.

For weaker properties, we do not have a complete list, but some results are available. In the case of primitive groups, these usually depend on the O’Nan–Scott classification of such groups, described in the topic essay on groups.

We give two examples taken from [1].

Theorem 4 *For regular permutation groups, the conditions "AS-friendly" and "stratifiable" are equivalent. A regular group has these properties if and only if either it is abelian or it is the direct product of the quaternion group of order* 8 *and an elementary abelian* 2*-group.*

Theorem 5 *An AS-free permutation group is primitive, and is either* 2*-homogeneous, or almost simple, or of diagonal type with at least four simple factors in its socle.*

In the last case, it is not known whether a group of diagonal type can be AS-free. There do exist almost simple groups which are AS-free but not 2-transitive; the smallest has degree 234.

Now let D be an incomplete-block design on a set Ω, with automorphism group G. If D is partially balanced, then the balancer of D is preserved by G.

If G is stratifiable, then obviously D is partially balanced with respect to the orbital scheme of G. Weaker conditions on G do not obviously imply any partial balance properties for D. This is a topic which needs more investigation!

References

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